Math 2050, alternative proof to uniform continuity theorem

Theorem 0.1. *Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function for some $a, b \in \mathbb{R}$, then f is uniform continuous.*

Proof. Let $\varepsilon > 0$ be given. We let S be the subset of $[a,b]$ containing $c \in [a,b]$ such that we can find $\delta > 0$ so that for all $x, y \in [a,c]$ with $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Clearly, $a \in S$ trivially. By completeness of real number, $s = \sup S$ exists. By continuity of f at $x = a$, we must have $s \in (a,b]$. We want to show that $s = b \in S$. If this is the case, then we can find $\delta > 0$ so that for all $x, y \in [a,b]$ with $|x-y| < \delta$, we will have

$$|f(x) - f(y)| < \varepsilon.$$

Then we are done.

Suppose $s < b$, since f is continuous at $x = s$ we can find $\delta_0 > 0$ such that for all $x, y \in (s - \delta_0, s + \delta_0) \subsetneq [a,b]$, we have

$$|f(x) - f(y)| \leq |f(x) - f(s)| + |f(y) - f(s)| < \varepsilon.$$

Since $s = \sup S$ and $a < s - \delta_0 < \sup S$, there exists $s_1 \in S$ such that $s - \frac{1}{2}\delta_0 < s_1 \leq \sup S$. In particular, we can find $\delta_1 > 0$ such that for all $x, y \in [a, s_1]$ with $|x - y| < \delta_1$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Now we claim that $s + \frac{1}{2}\delta_0 \in S$. For all $x, y \in [a, s + \frac{1}{2}\delta_0]$ where $|x - y| < \delta_2 = \min\{\delta_1, s_1 - s + \frac{1}{2}\delta_0\}$, we have either $x, y \in [a, s_1]$ or $x, y \in [s - \frac{1}{2}\delta_0, s + \frac{1}{2}\delta_0]$ since $|x - y| < s_1 - s + \frac{1}{2}\delta_0$ (draw the graph to visualize it).

In the first case, since $|x - y| < \delta_1$ we have

$$|f(x) - f(y)| < \varepsilon.$$

In the second case, since $x, y \in (s - \delta_0, s + \delta_0)$, we also have

$$|f(x) - f(y)| < \varepsilon.$$

That said, $s + \frac{1}{2}\delta_0 \in S$ implying contradiction. Hence $s = b$. Moreover, taking $s = b$ in the above argument, we see that $s = b \in S$.

$\square$